GIRRAWEEN HIGH SCHOOL

HSC TASK 3 2025

MATHEMATICS

EXTENSION 2

Time allowed – 90 Minutes (plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- NESA-approved calculators may be used. Reference sheets will be provided.
- Each question attempted is to be marked clearly Question 6, Question 7 etc. Each question is to be attempted on a separate page in your answer booklet.
- You may ask for spare answer booklets if you need them.
- For Multiple choice: Fill in the circle corresponding to the correct answer on the multiple choice answer sheet in your answer booklet.
- For Questions 6-11: All necessary working should be shown in each question. Marks may be deducted for careless or badly arranged work.

Multiple Choice begins on the following page

For Multiple choice Questions 1-5: Fill in the circle corresponding to the correct answer on the multiple choice answer sheet in your answer booklet.

Question 1

$$\int \frac{x+1}{(x-2)(x-1)}.dx =$$

(A) $ln\left(\frac{(x-2)^2}{(x-1)^3}\right) + C$

(B) $ln\left(\frac{(x-1)^2}{(x-2)^3}\right) + C$

(C) $ln\left(\frac{(x-2)^3}{(x-1)^2}\right) + C$

(D) $ln\left(\frac{(x-1)^3}{(x-2)^2}\right) + C$

Question 2

$$\int_0^3 \frac{x}{\sqrt{x+1}}.dx =$$

(A) $\frac{2}{3}$

(B) $\frac{4}{3}$

(C) 2

(D) $\frac{8}{3}$

Multiple Choice continues on the following page

Question 3

$$\int_0^a f(2a - x).dx =$$

(A) $\int_a^{2a} f(x).dx$

(B) $\int_a^{2a} f(2a - x).dx$

(C) $\int_a^{2a} f(x).dx$

(D) $\int_{-a}^{a} f(x).dx$

Question 4

A particle moving with Simple Harmonic Motion is moving according to the formula $v = 6\cos 3t$, where v is the velocity in m/s and t is the time in seconds. The period and amplitude of its motion are:

(A) Period = $\frac{2\pi}{3}$ seconds, amplitude = $2m$

(B) Period = $\frac{2\pi}{3}$ seconds, amplitude = $6m$

(C) Period = $\frac{2\pi}{3}$ seconds, amplitude = $18m$

(D) Period = 3 seconds, amplitude = $6m$

Multiple Choice continues on the following page

Question 5

The displacement of a particle x metres from the point $x = 0$ is given by $x = v^4 + 1, v > 0$.

where v is the velocity in m/s. The acceleration of the particle is given by:

(A) $\ddot{x} = 12v^2$

(B) $\ddot{x} = \sqrt[4]{x-1}$

(C) $\ddot{x} = \frac{1}{4\sqrt{x-1}}$

(D) $\ddot{x} = \frac{1}{\sqrt[3]{t^2}}$

Question 6 *(11 marks) (Start solutions on a new page)* **Marks**

(a) $\int tan^3x\ sec^4x.dx =$ **2**

(b) $\int \frac{x^2-1}{x^2+2x+5}.dx =$ **2**

(c) $\int xe^{2x}.dx =$ **2**

(d) Use the substitution $t = tan\left(\frac{x}{2}\right)$ to find **5**

$$\int \frac{4}{3\sin x + 2\cos x + 3}.dx$$

Question 7 *(13 Marks) (Start solutions on a new page)* **Marks**

(a) (i) Find real numbers A, B and C such that 2

$$\frac{-7x-9}{(x^2+1)(x-3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3}$$

(ii) Hence find $\int \frac{-7x-9}{(x^2+1)(x-3)}.dx$ 2

(b) (i) By letting $x = 2\sec\theta$, show that 3

$$\int \frac{1}{\sqrt{x^2-4}}.dx = \ln\left(x+\sqrt{x^2-4}\right) + C$$

(ii) Hence or otherwise find $\int \sqrt{\frac{x}{x+4}}.dx$ 4

(c) Find $\int sin^{-1}(x).dx$ 2

EXAMINATION CONTINUES ON THE FOLLOWING PAGE

Question 8 *(15 Marks) (Start solutions on a new page)* **Marks**

(a) A particle moves according to the rule $x = 3\sin\left(2t + \frac{\pi}{3}\right)$ where x is in metres from $x = 0$ and t is the time in seconds from $t = 0$.

(i) Show that the particle is moving in Simple Harmonic Motion. 1

(ii) State the period and amplitude of the particle's motion. 2

(iii) Find the particle's initial position, velocity and acceleration. 3

(iv) Find the first two times when the particle will reach $x = 1.5$ and find its velocity on both occasions. 4

(v) Find the particle's maximum speed and find the first time this will happen. 2

(b) A particle moves with SHM so that it follows the rule $x = a\sin(nt + \alpha)$. 3

If the particle's period is $\frac{2\pi}{3}$ seconds and initially it is at $x = 1$ with velocity $3\sqrt{3}\ m/s$, find the value of a, n and α.

EXAMINATION CONTINUES ON THE FOLLOWING PAGE

Question 9 *(14 Marks) (Start solutions on a new page)* **Marks**

(a) A particle moves according to the rule $v^2 = -9x^2 + 72x - 108$
where x is in metres and v is in metres per second.

(i) Show that the particle is moving in simple harmonic motion. **1**

(ii) Find the centre of motion, period and amplitude of the particle's motion. **4**

(b) A bullet is shot from a gun at an initial velocity of $500m/s$. It experiences acceleration of $-\frac{v^2}{360}\ m/s^2$.

(i) Taking the starting point as $x = 0$, show that $v = 500\ e^{-\frac{x}{360}}$. **2**

(ii) The bullet ceases to be lethal once its speed drops below $100m/s$. **2**
How far will it have travelled when this happens (answer to the nearest metre).

(iii) Show that the time taken for the bullet to travel a distance of x metres **2**
is given by $t = \frac{18}{25}\left(e^{\frac{x}{360}} - 1\right)$

(iv) Express time in terms of velocity. Hence find how long the bullet is **3**
in the air before its speed drops below $100m/s$.

EXAMINATION CONTINUES ON THE FOLLOWING PAGE

Question 10 *(12 Marks) (Start solutions on a new page)* **Marks**

(a) If $u_1 = 6$ and $u_N = 5u_{N-1} - 4$, prove by induction that $u_N = 5^N + 1$ **3**

(b) (i) Explain why, $\forall$ positive integers n, $n(n+1)$ is even. **1**

(ii) Use induction to prove that $n^3 - 4n$ is divisible by 48 **3**
$\forall$ positive even integers $n \geq 4$ *(you may use (i) in your proof).*

(c) (i) Solve the inequality $4n^2 - 2n - 1 > 0$ **2**

(ii) Use mathematical induction to prove $5^n > n^2$ *(you may use the result from part (i) in your proof).* **3**

Question 11 *(11 Marks) (Start solutions on a new page)* **Marks**

(a) If $I_N = \int tan^N x.\,dx,$

(i) Show that $I_N = \frac{tan^{N-1}x}{N-1} - I_{N-2}$ 2

(ii) Hence find I_6 3

(b) If $I_N = \int_0^{\frac{\pi}{4}} x^N \cos x.\,dx$

(i) Show that $I_N = \frac{\pi^{N-1}\sqrt{2}(\pi+4N)}{4^N \times 2} - N(N-1)I_{N-2}$ 3

(ii) Hence find I_4 3

END OF EXAMINATION

Y12 Extension 2 Task 3 2025 Solutions:

Multiple Choice: (1) C (2) D (3) C (4) A (5) C

(1) $\frac{x+1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$

$\therefore x+1 = A(x-1) + B(x-2)(1)$

Sub. $x=1$ in (1): $2 = -B \Rightarrow B = -2$.

Sub. $x=2$ in (1): $3 = A$.

$\therefore \int \frac{x+1}{(x-2)(x-1)} \cdot dx = \int \frac{3}{x-2} - \frac{2}{x-1} \cdot dx$

$= \ln\left[\frac{(x-2)^3}{(x-1)^2}\right] + C$ (C)

(2) $\int_0^3 \frac{x}{\sqrt{x+1}} \cdot dx$

$= \int_0^3 \frac{x+1}{\sqrt{x+1}} - \frac{1}{\sqrt{x+1}} \cdot dx$

$= \int_0^3 \sqrt{x+1} - \frac{1}{\sqrt{x+1}} \cdot dx$

$= \left[\frac{2}{3}(x+1)\sqrt{x+1} - 2\sqrt{x+1}\right]_0^3$

$= \frac{8}{3}$ (D)

(3) $\int_0^a f(2a-x) \cdot dx$ Let $u = 2a-x$, $du = -1 \cdot dx$

$= \int_a^0 f(2a-x) \cdot -1 \cdot dx$

$= \int_a^{2a} f(u) \cdot du$

$= \int_a^{2a} f(x) \cdot dx$ (C) or (A)

Q.(4) $v = 6\cos 3t$.

$x = \int 6\cos 3t \, dt$

$= 2\sin 3t + C$.

Period $= \frac{2\pi}{3}$. Amplitude $= 2$m (A)

Q.(5) Either:

$x = v^4 + 1$. (note: $v = \sqrt[4]{x-1}$]. & $v^2 = \sqrt{x-1}$.

$\frac{dx}{dv} = 4v^3$.

$\frac{dv}{dx} = \frac{1}{4v^3}$

$\ddot{x} = v \cdot \frac{dv}{dx} = \frac{1}{4v^2}$

$= \frac{1}{4\sqrt{x-1}}$ (C)

(or)

$x = v^4 + 1$.

$v = \sqrt[4]{x-1}$.

$v^2 = \sqrt{x-1}$.

$\frac{d}{dx}(v^2) = \frac{1}{2\sqrt{x-1}}$.

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{4\sqrt{x-1}}$ = (C) again.

Q.(6)(a) $\int \tan^3 x \sec^4 x \, dx$

$= \int \tan^3 x . \sec^2 x . \sec^2 x \, dx$

$= \int \tan^3 x . (\tan^2 x + 1) . \sec^2 x \, dx$

$= \int (\tan^5 x + \tan^3 x) . \sec^2 x \, dx$

$= \int u^5 + u^3 \, du$

$= \frac{1}{6} u^6 + \frac{1}{4} u^4 + C$

$= \frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + C.$

Note: Can also use $\int [f(x)]^n . f'(x) . dx$ rule.

$u = \tan x$.
$du = \sec^2 x . dx$

Mostly well done.

2

(b) $\int \frac{x^2 - 1}{x^2 + 2x + 5} \, dx$

$= \int \frac{x^2 + 2x + 5}{x^2 + 2x + 5} - \frac{2x + 2}{x^2 + 2x + 5} - \frac{4}{x^2 + 2x + 5} \, dx$

$= \int 1 - \frac{2x + 2}{x^2 + 2x + 5} - \frac{4}{(x+1)^2 + 4} \, dx$

$= x - \ln(x^2 + 2x + 5) - 2 \tan^{-1}\left(\frac{x+1}{2}\right) + C$

Don't actually need absolute value in integration as $x^2 + 2x + 5 > 0$ $\forall$ real x.

Marker's comments:

(a) Mostly well done.
Students could also use $u = \sec x$, $du = \sec x \tan x \, dx$ & get $\frac{1}{6} \sec^6 x - \frac{1}{4} \sec^4 x + C$.

(b) VERY poorly done. Students did NOT divide the rational function up into the 3 parts before integration!

Q.(6)(c) $\int xe^{2x}.dx$ $\quad u=x \quad v=\frac{1}{2}e^{2x}$

$u'=1. \quad v'=e^{2x}$

By $\int uv'.dx = uv - \int vu'.dx$

Generally well done.

$\int xe^{2x}.dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}.dx$

$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$

(d) $\int \frac{4}{3\sin x + 2\cos x + 3}.dx \quad t=\tan\left(\frac{x}{2}\right)$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

$= \frac{t^2+1}{2}.$

$\therefore dx = \frac{dx}{dt}.dt = \frac{2}{t^2+1}.dt$

$= \int \frac{4\times 2}{\frac{3\times 2t}{1+t^2} + \frac{2\times(1-t^2)}{1+t^2} + 3 + 3t^2}.dt$

$= \int \frac{8}{t^2+6t+5}.dt$

$= \int \frac{8}{(t+2)(t+3)}.dt$ $\quad$ Let $\frac{8}{(t+2)(t+3)} = \frac{A}{(t+2)} + \frac{B}{(t+3)}$

$8 = A(t+3)+B(t+2)$

$t=-3: 8=-B \Rightarrow B=-8.$

$t=-2: 8=A.$

$= \int \frac{8}{t+2} - \frac{8}{t+3}.dt$

$= 8\ln\left[\frac{t+2}{t+3}\right] + C$

$= 8\ln\left[\frac{\tan\left(\frac{x}{2}\right)+2}{\tan\left(\frac{x}{2}\right)+3}\right] + C.$

Marker's comments:

(6)(c) Generally well done.

(d) Most students could get to $\int \frac{8}{t^2+6t+5}.dt$, but then instead of factorising t^2+6t+5 & using partial fractions, did some VERY strange things, including completing the square.

Note: Some students then went on to use $x=\sec\theta$ as a substitution. DON'T DO THIS UNLESS IT'S UNDER A SQUARE ROOT SIGN!

Q.(7)(a)(i) $\dfrac{-7x-9}{(x^2+1)(x-3)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-3}$

Note: Using "cover up" rule Simultaneous equations or substituting $x=i$ are all acceptable here.

$\therefore -7x-9 = (Ax+B)(x-3) + C(x^2+1)$

Sub. in $x=3$: $-30 = 10C \Rightarrow C=-3$.

Sub. in $x=0, C=-3$: $-9 = -3B - 3 \Rightarrow B=2$

Sub. in $x=1, B=2, C=-3$: $-16 = -2(A+2) + -3\times 2 \Rightarrow A=3$. 2

Hence $\dfrac{-7x-9}{(x^2+1)(x-3)} = \dfrac{3x+2}{x^2+1} - \dfrac{3}{x-3}$.

Marks.

Generally well done.

(ii) Hence $\int \dfrac{-7x-9}{(x^2+1)(x-3)}\,dx$

$= \int \dfrac{3x+2}{x^2+1} - \dfrac{3}{x-3}\,dx$

$= \int \dfrac{3}{2}\times\dfrac{2x}{x^2+1} + \dfrac{2}{x^2+1} - \dfrac{3}{x-3}\,dx$

$= \dfrac{3}{2}\ln(x^2+1) + 2\tan^{-1}(x) - 3\ln|x-3| + C.$

(b)(i) $\int \dfrac{1}{\sqrt{x^2-4}}\,dx \quad x = 2\sec\theta.\ dx = 2\sec\theta\tan\theta\,d\theta$

Marks.

Generally well done.

$= \int \dfrac{1}{2\tan\theta}\, 2\sec\theta\tan\theta\,d\theta$

$= \int \sec\theta\,d\theta$

$= \int \dfrac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}\,d\theta$

$= \int \dfrac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta$

] must show this step

$= \ln|\sec\theta+\tan\theta| + C = \ln\left|\dfrac{x}{2} + \dfrac{\sqrt{x^2-4}}{2}\right|$

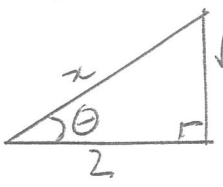
$= \ln\left|x+\sqrt{x^2-4}\right| - \ln 2 + C$ → must show this step to get full marks

$= \ln\left|x+\sqrt{x^2-4}\right| + C$. (as $\ln 2$ constant).

Note: If $x = 2\sec\theta$,

$\dfrac{x}{2} = \sec\theta$

$\dfrac{2}{x} = \cos\theta$



$\tan\theta = \dfrac{\sqrt{x^2-4}}{2}$

$\therefore \dfrac{2}{x}\tan\theta = \sqrt{x^2-4}$.

Q.(7)(b)(ii) $\int \sqrt{\frac{x}{x+4}}\,dx$ $\times\sqrt{x}$ $\times\sqrt{x}$

Poorly done many did not know the technique required.

$= \int \frac{x}{\sqrt{x(x+4)}}\,dx$

$= \int \frac{x+2}{\sqrt{x^2+4x}}\,dx - \int \frac{2}{\sqrt{x^2+4x}}\,dx$ — This step poorly done.

$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+4x}}\,dx - \int \frac{2}{\sqrt{(x+2)^2-4}}\,dx$ ↙ Using (i)

$= \sqrt{x^2+4x} - 2\ln\left|(x+2)+\sqrt{x^2+4x}\right| + C.$ ↗

4

Using $\int f'(x)(f(x))^n\,dx = \frac{1}{n+1}(f(x))^{n+1}$

Q.(7)(c) $\int \sin^{-1}(x)\,dx$ $\quad u = \sin^{-1}(x)$ $\quad v = x$

$u' = \frac{1}{\sqrt{1-x^2}}$ $\quad v' = 1$

2

By $\int uv'\,dx = uv - \int vu'\,dx$

$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$

Generally well done.

$= x\sin^{-1}(x) + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$

$= x\sin^{-1}(x) + \sqrt{1-x^2} + C.$

Marker

Q.(8)(a)(i) $x = 3\sin\left(2t+\frac{\pi}{3}\right)$

$\dot{x} = 6\cos\left(2t+\frac{\pi}{3}\right)$

$\ddot{x} = -12\sin\left(2t+\frac{\pi}{3}\right)$ 1

$= -4x.$

done well

As $\ddot{x} = -4x$, particle is moving with SHM.

(ii) Period $= \frac{2\pi}{2} = \pi$ seconds. 2

Amplitude $= 3$m.

done well

(iii) Initial position: $x = 3\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2} \to \frac{3\sqrt{3}}{2}$ m RIGHT of $x=0$ ($=2{\cdot}6$m).

Initial velocity: $\dot{x} = 6\cos\frac{\pi}{3} = 3$m/s. 3

Initial acceleration $\ddot{x} = -12\sin\frac{\pi}{3} = -6\sqrt{3}$ m/s^2 $= -10{\cdot}4$m/s^2.

or $\ddot{x} = -4x = -4\times\frac{3\sqrt{3}}{2}$

done well

(iv) $x = 1{\cdot}5$. $3\sin\left(2t+\frac{\pi}{3}\right) = 1{\cdot}5$

$\sin\left(2t+\frac{\pi}{3}\right) = 0{\cdot}5$

$2t+\frac{\pi}{3} = \frac{5\pi}{6}, \frac{13\pi}{6}$ [not $\frac{\pi}{6}$ as if $2t+\frac{\pi}{3}=\frac{\pi}{6}$, $t=\frac{-\pi}{12}$]. 4

$t = \frac{\pi}{4}$ seconds or $\frac{11\pi}{12}$ seconds.

Instead of $\frac{11\pi}{12}$, some student got

When $t=\frac{\pi}{4}$, $\dot{x} = 6\cos\frac{5\pi}{6} = -\frac{6\sqrt{3}}{2} = -3\sqrt{3}$ m/s.

When $t=\frac{11\pi}{12}$, $\dot{x} = 6\cos\frac{13\pi}{6} = 3\sqrt{3}$ m/s.

Other answers.

Marker

(v) Max. SPEED occurs where $\cos\left(2t+\frac{\pi}{3}\right) = \pm 1$.

If $\cos\left(2t+\frac{\pi}{3}\right) = 1$. $2t+\frac{\pi}{3} = 0, 2\pi$.

t negative for $2t+\frac{\pi}{3}=0$

If $\cos\left(2t+\frac{\pi}{3}\right) = -1$. $2t+\frac{\pi}{3} = \pi$. $t = \frac{\pi}{3}$.

Max. Speed $= |6\cos\pi| = |-6| = 6$ m/s. 2

after $\frac{\pi}{3}$ seconds.

Some didn't write maximum speed. This is confusing for many students.

Q.(8)(b) Period $= \frac{2\pi}{3}$.

$\frac{2\pi}{n} = \frac{2\pi}{3} \Rightarrow n = 3.$

$x = a\sin(3t + \alpha)$

$x = 1$ when $t = 0 \Rightarrow a\sin\alpha = 1$ (1)

$\dot{x} = 3a\cos(3t + \alpha)$

$\dot{x} = 3\sqrt{3}$ when $t = 0 \Rightarrow 3a\cos\alpha = 3\sqrt{3}$

$a\cos\alpha = \sqrt{3}$ (2)

3

(1) ÷ (2) $\tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}.$

Sub. $\alpha = \frac{\pi}{6}$, $n = 3$ in (1): $a\sin\frac{\pi}{6} = 1.$

$\frac{a}{2} = 1.$

$a = 2.$

done well.

$\therefore a = 2, n = 3, \alpha = \frac{\pi}{6}.$

$x = 2\sin(3t + \frac{\pi}{6})$

Marker's Comments

(a)(b)(i) $v^2 = -9x^2 + 72x - 108$

$\frac{d}{dx}(v^2) = -18x + 72$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9x + 36.$

$\ddot{x} = -9(x-4).$

As $\ddot{x} = -n^2(x-C)$, particle is moving with SHM.

done well.

(ii) Centre of motion: $x = 4$.

Period: $n^2 = 9 \Rightarrow n = 3$. Period $= \frac{2\pi}{3}$.

Amplitude: If $v = 0$: $-9x^2 + 72x - 108 = 0$

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0.$

Stops at $x = 6$ & $x = 2 \Rightarrow$ centre $x = 4$.

done well

Amplitude = 2 metres

$\therefore$ Period $= \frac{2\pi}{3}$, amplitude = 2m, centre of motion $x = 4$.

Note: Will also accept $v^2 = 9[4 - (x-4)^2]$

by $v^2 = n^2[a^2 - (x-C)^2]$.

Marker's Comments

Q.(9)(b)(i) $\ddot{x} = v\frac{dv}{dx} = \frac{-v^2}{360}$

$\frac{dv}{dx} = \frac{-v}{360}$

done well.

$\frac{dx}{dv} = \frac{-360}{v}$

$x = \int \frac{-360}{v}\, dv$ 2

$x = -360\ln v + C$

If $v=500$ when $x=0$, $0 = -360\ln 500 + C \Rightarrow C = 360\ln 500$

$x = -360\ln\left(\frac{v}{500}\right)$ [Note: No need for absolute value here as once $v=0$, $\ddot{x}=0$ & all motion stops → no negative velocity].

$v = 500e^{-\frac{x}{360}}$

(ii) Finding x when $v = 100$. $100 = 500e^{-\frac{x}{360}}$ done well

$\frac{1}{5} = e^{-\frac{x}{360}}$ 2

$\ln\left(\frac{1}{5}\right) = -\frac{x}{360}$

$x = 579.4\text{m}.$

Marker's Comments.

Q.(9)(b)(iii) $v = \frac{dx}{dt} = 500e^{-\frac{x}{360}}$

$$\frac{dt}{dx} = \frac{1}{500} e^{\frac{x}{360}}$$

$$t = \frac{1}{500}\int e^{\frac{x}{360}} \cdot dx$$

$$= \frac{18}{25} e^{\frac{x}{360}} + C$$

2

$t=0, x=0: 0 = \frac{18}{25} e^{0} + C \Rightarrow C = -\frac{18}{25}$.

$$t = \frac{18}{25}\left[e^{\frac{x}{360}} - 1\right].$$

done well.

$= 0.107..$ seconds.

(iv). $\ddot{x} = \frac{dv}{dt} = -\frac{v^2}{360}$

$$\frac{dt}{dv} = -\frac{360}{v^2}$$

$$t = \int -\frac{360}{v^2} \cdot dv$$

$$= \frac{360}{v} + C$$

Note: Had to use HENCE here rather than using the answers to (iii) & (iv)

3

$t=0 \Rightarrow v=500$ $0 = \frac{360}{500} + C \Rightarrow C = -\frac{18}{25}$.

$$t = \frac{360}{v} - \frac{18}{25}.$$

If $v = 100$, $t = \frac{360}{100} - \frac{18}{25} = 2.88$ seconds.

Marker's comments

To get $t = \frac{360}{v} - \frac{18}{25}$ students used (i) and (iii). They were given mark.

Q.(10)(a) Step 1: Show true for $n=1$.

LHS
$= 5^1 + 1$
$= 6$
$= u_1$

True for $n=1$.

Step 2: Assume true for $n=k$
i.e. $u_k = 5^k + 1$.

Step 3: Prove true for $n = k+1$
i.e. RTP $u_{k+1} = 5^{k+1} + 1$

LHS

u_{k+1}
$= 5u_k - 4$
$= 5(5^k + 1) - 4$ from assumption
$= 5^{k+1} + 5 - 4$
$= 5^{k+1} + 1$
$=$ RHS QED.

If true for $n=k$ it will be true for $n=k+1$.
Hence it will be true $\forall n \in \mathbb{Z}^+$ by the principle of mathematical induction.

Poorly done overall.
many forgot to write "from assumption"
This was penalised once in Q10.

Some students started with $u_{k+1} = 5^{k+1} + 1$
but this is what you had to prove.
so no further marks awarded after step!

Marker's comments

Q.(10)(b)(i) If n is odd then $n = 2k-1$

$n+1 = 2k$

$n(n+1) = (2k-1)2k$ which is even.

If n is even, then $n = 2k$

$n+1 = 2k+1$.

$n(n+1) = 2k(2k+1)$ which is even.

Poorly done. many just stated that the product of an even and an odd number is even but this did not answer the question.

(ii) Step 1: Show true for $n = 4$

LHS

$= 4^3 - 4 \times 4$

$= 48$

which is divisible by 48.

Step 2: Assume true for $n = 2k$, $k \in Z^+$ [as n is even].

i.e. $(2k)^3 - 4 \times 2k = 48P$, $P \in Z^+$

$8k^3 - 8k = 48P$

$8k(k-1)(k+1) = 48P$.

Step 3: Prove true for $n = 2k+2$ 3

i.e. $[2(k+1)]^3 - 4 \times 2(k+1)] = 48Q$, $Q \in Z^+$

LHS

$[2(k+1)]^3 - 4 \times 2(k+1)$

$= 8(k+1)^3 - 8(k+1)$

$= 8[(k+1)(k)(k+2)]$

$= 8k(k+1)[k-1+3]$

$= 8k(k+1)(k-1) + 24k(k+1)$

$= 48P + 24 \times 2R$ [By assumption & using part (i)]

$= 48[P+R]$

which is divisible by 48.

Students who assumed true for $n = k$, then tried to prove true for $n = k+1$ were not successful.

If it is true for $n = 2k$ it will be true for $n = 2(k+1)$. Hence it will be true $\forall$ even $n \in Z^+$ by the principle of mathematical induction.

MARKER'S COMMENTS.

Poorly done overall.

See alternate solution on next page.

(10)(6) Alternative proof for Step 2 onwards

Step 2: Assume true for $n=k$, k even.

i.e. $k^3 - 4k = 48P$, $P \in \mathbb{Z}^+$.

Step 3: Prove true for $n = k+2$ (as $k+2$ is next even number)

i.e. $(k+2)^3 - 4(k+2) = 48Q$, $Q \in \mathbb{Z}^+$.

LHS

$$(k+2)^3 - 4(k+2)$$
$$= (k+2)[(k+2)^2 - 4]$$
$$= (k+2)k(k+4)$$
$$= k(k+2)[(k-2)+6]$$
$$= k(k+2)(k-2) + 6k(k+2)$$
$$= 48P + 6k(k+2) \quad (1) \text{ (By assumption}$$

Taking $6k(k+2)$ out of (1).

If k, $k+2$ even.

∴ one of k or $k+2$ is divisible by 4.

∴ $6k(k+2)$ must have a factor $6 \times 4 \times 2 = 48$.

∴ $6k(k+2) = 48Q$, $Q \in \mathbb{Z}^+$. (2)

Sub. (2) in (1):

$$= 48P + 48Q$$
$$= 48(P+Q)$$

which is divisible by 48.

Q.(10)(c) (i) $4n^2-2n-1>0$.

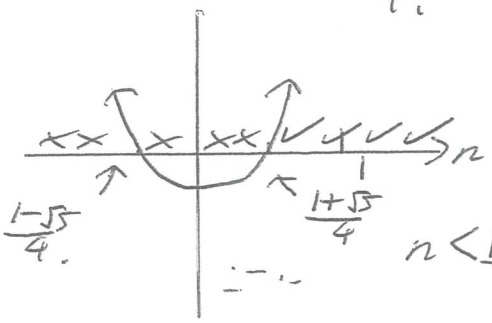
Surprisingly, poorly done.

Finding where $4n^2-2n-1=0$

$$n=\frac{2\pm\sqrt{(-2)^2-4\times4\times-1}}{2\times4}$$

$$n=\frac{1\pm\sqrt{5}}{4}.$$

Students should show a diagram or how they got their solution. Not penalised, this time.



$\therefore$ $n<\frac{1-\sqrt{5}}{4}$ or $n>\frac{1+\sqrt{5}}{4}$.

(ii) Step 1: show true for $n=1$.

LHS	RHS	
$=5^1$	$=1^2$	True for $n=1$.
$=5$.	$=1$.	

Step 2: Assume true for $n=k$

i.e. $5^k>k^2$, $k\in\mathbb{Z}^+$

Step 3: Prove true for $n=k+1$

i.e. $5^{k+1}>(k+1)^2$, $k\in\mathbb{Z}^+$

LHS

5^{k+1}

$=5\times5^k$

$>5k^2$ [by assumption]

$>(k+1)^2$ [as $5k^2>(k+1)^2$ because $4k^2-2k-1>0$ using (i)].

$\therefore 5^{k+1}>5k^2>(k+1)^2$

$5^{k+1}>(k+1)^2$.

If it is true for $n=k$ it will be true for $n=k+1$.

Hence as it is true for $n=1$ it will be true $\forall n\in\mathbb{Z}^+$

(or) $5^{k+1}-(k+1)^2>0$

LHS

$5^{k+1}-(k+1)^2$

$=5\times5^k-(k+1)^2$

$>5k^2-(k+1)^2$ [By assumption]

$=4k^2-2k-1$

>0 [using Part (i)].

Maker's Comments

from part (i)

$4k^2-2k-1>0$

$4k^2>2k+1$

$5k^2>k^2+2k+1$

$5k^2>(k+1)^2$

Q.(11)(a)(i)

$$I_N = \int \tan^N x \, . \, dx$$

$$= \int \tan^{N-2} x \, . \, \tan^2 x \, . \, dx$$

$$= \int \tan^{N-2} x \, (\sec^2 x - 1) \, . \, dx$$

$$I_N = \int \tan^{N-2} x \sec^2 x \, . \, dx - \int \tan^{N-2} x \, . \, dx$$

$$I_N = \frac{1}{N-1} \tan^{N-1} x - I_{N-2}$$

2

(ii) $I_0 = \int \tan^0 x \, . \, dx$

$$= \int 1 \, . \, dx$$

$$= x \, .$$

$$I_2 = \tan x - x \, .$$

$$I_4 = \frac{1}{3} \tan^3 x - (\tan x - x)$$

$$= \frac{1}{3} \tan^3 x - \tan x + x \, .$$

$$I_6 = \frac{1}{5} \tan^5 x - \left(\frac{1}{3} \tan^3 x - \tan x + x \right)$$

$$I_6 = \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \tan x - x + C$$

3

Marker's comments

(a)(i) Generally well done, BUT about 20 students forgot that $\int \tan^n x \, . \, dx$ is one of the rare occasions you DON'T use integration by parts to find a reduction formula.

(ii) was generally well done.

Q.(ii)(b) (i) $I_N = \int_0^{\frac{\pi}{4}} x^N \cos x \, dx$ P17

$u = x^N \quad v = \sin x$

$u' = Nx^{N-1} \quad v' = \cos x$

By $\int uv' \, dx = uv - \int vu' \, dx$

$$I_N = \int_0^{\frac{\pi}{4}} x^N \cos x \, dx = \left[x^N \sin x\right]_0^{\frac{\pi}{4}} - N\int_0^{\frac{\pi}{4}} x^{N-1} \sin x \, dx \quad (1)$$

3 to here.

Taking $\int_0^{\frac{\pi}{4}} x^{N-1} \sin x \, dx$ out of (1): $u = x^{N-1} \quad v = -\cos x$

$u' = (N-1)x^{N-2} \quad v' = \sin x$

$$\int_0^{\frac{\pi}{4}} x^{N-1} \sin x \, dx = \left[-\cos x \, x^{N-1}\right]_0^{\frac{\pi}{4}} + (N-1)\int_0^{\frac{\pi}{4}} x^{N-2} \cos x \, dx$$

$$= -\frac{\pi^{N-1}\sqrt{2}}{2 \times 4^{N-1}} + (N-1) I_{N-2} \quad (2)$$

Sub. (2) in (1):

3

$$I_N = \left[x^N \sin x\right]_0^{\frac{\pi}{4}} + N\left[\frac{\pi^{N-1}\sqrt{2}}{2 \times 4^{N-1}}\right] - N(N-1) I_{N-2}$$

$$= \frac{\pi^N \sqrt{2}}{2 \times 4^N} + \frac{N \times \pi^{N-1}\sqrt{2}}{2 \times 4^{N-1}} - N(N-1) I_{N-2}$$

$$= \frac{\pi^N \sqrt{2} + 4N\pi^{N-1}\sqrt{2}}{2 \times 4^N} - N(N-1) I_{N-2}$$

$$I_N = \frac{\pi^{N-1}\sqrt{2}[\pi + 4N]}{2 \times 4^N} - N(N-1) I_{N-2}.$$

as required.

1 ≤ cot ≤ tan > cos P.10 19 (or) K.18

Q.(ii)(b)(iii) $I_0 = \int_0^{\frac{\pi}{4}} \cos x \, dx$

$= [\sin x]_0^{\frac{\pi}{4}}$

$= \frac{\sqrt{2}}{2}.$

$I_2 = \frac{\pi\sqrt{2}(\pi+8)}{2\times 4^2} - 2\times 1\times\frac{\sqrt{2}}{2}$

$= \frac{(\pi^2+8\pi-32)\sqrt{2}}{32}$ 3

$I_4 = \frac{\pi^3\sqrt{2}(\pi+16)}{2\times 4^4} - \frac{4\times 3\times[\pi^2+8\pi-32]\sqrt{2}}{32}$

$= \frac{\pi^4\sqrt{2}+16\pi^3\sqrt{2}-192\pi^2\sqrt{2}-1536\pi\sqrt{2}+6144\sqrt{2}}{512}$

$= \frac{\sqrt{2}(\pi^4+16\pi^3-192\pi^2-1536\pi+6144)}{512}$

$\doteq 0.0471.$ (4 DP).

Maher's comments.

Using $I_n = \frac{\pi^{N-1}\sqrt{2}(\pi+4N)}{2\times 4^N} - N(N-1)I_{N-2}.$

8 If original (correct) formula was used.

END OF SOLUTIONS.